lec01

2025-03-17

Teoria dos tipos

Formalização de matemática

Fundamentos de matemática

Teoria dos conjuntos

Teoria das categorias

Matemática construtiva

Teoria das demonstrações

Lógica matemática (FOL, ...)

Teoria dos grupos, aneis, ...

Teoria dos inteiros, reais, ...

Geometria euclideana

λ-calculus

Programação funcional

Proof assistants

Theorem provers

static-vs-dynamic

axiomas-vs-leis

Zermelo - Euclides

Representando números como conjuntos

Zermelo	von Neumann
$0 \equiv \{\} \equiv \emptyset$	$\{\}$
$1 \equiv \{\{\}\}$	$\{0\}$
$2 \equiv \{\{\{\}\}\}$	$\{0, 1\}$
$\vdots$	$\vdots$

$$n \mid m \overset{\text{def}}{\iff} (\exists k \in \mathbb{Z})[n \cdot k = m]$$

$$A \subseteq B \overset{\text{sug}}{\iff} (\forall x)[x \in A \Rightarrow x \in B]$$

$0 \in 2$?

$3 \subseteq 5$? — essas perguntas deveriam mesmo ser perguntáveis?

OK sobre os $0, 1, 2, \ldots$ mas cadê o $\mathbb{N} = \{0, 1, 2, \ldots\}$?

Representando tuplas como conjuntos

$$\langle 7, 2 \rangle \overset{\text{imp}}{\equiv} \{7, 2\}$$

$$\langle 7, 2 \rangle \overset{\text{imp}}{\equiv} \{\{7\}, \{7,2\}\}$$

$$A \times B \overset{\text{imp}}{\equiv} ?$$

(TUP1) $\langle x, y \rangle = \langle x', y' \rangle \iff x = x' \ \& \ y = y'$

(TUP2) $\{ \langle a, b \rangle \mid a \in A, b \in B \}$ deve ser um conjunto

Sistemas Hilbert

?: $\alpha \supset (\beta \supset \alpha)$

?: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

?: $(\neg\alpha \supset \neg\beta) \supset (\beta \supset \alpha)$

Como inferir: (Modus Ponens)

Tendo $\alpha \supset \beta$ e α inferir β

$\vdash A \to A$

Gentzen

Natural deduction

Sequent calculus

Natural deduction

$$\frac{A \ \text{true} \qquad B \ \text{true}}{A \& B \ \text{true}}\ \&\text{-I} \qquad \frac{A \& B \ \text{true}}{A \ \text{true}}\ \&\text{-E}_1 \qquad \frac{A \& B \ \text{true}}{B \ \text{true}}\ \&\text{-E}_2$$

$$\frac{A \ \text{prop} \qquad B \ \text{prop}}{A \& B \ \text{prop}}$$

$$\left.\frac{\dfrac{A \& B}{B} \qquad \dfrac{A \& B}{A}}{B \& A}\right\}\ \text{comutatividade de } (\&)$$

$(\in)$ vs $(:)$, dynamic vs static, ...

```
x = ...
if (x é um int):

else:

```

compila

```
int x;
if ( o inteiro x ):

else:

```

não compila

sets

types

escrevemos $a \in A$

escrevemos $a : A$

Set theory ops/construções:

$\emptyset$, $\{\_,\_\}$, $\bigcup\_$, $\wp\_$, $\bigcap\_$

Type theory ops/construções:

(outras construções)

Internal ops vs external constructions lec04

2025-03-26

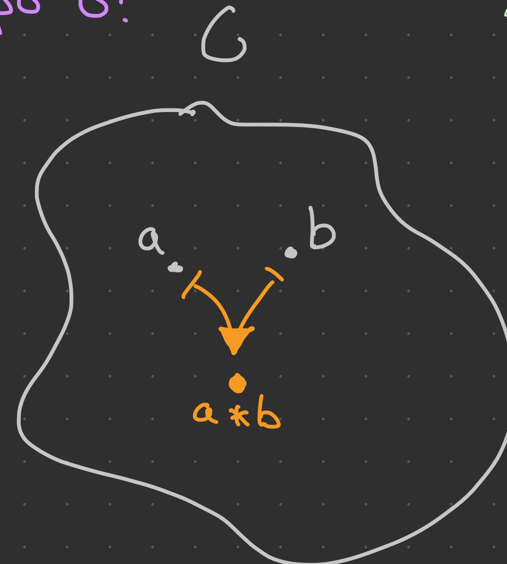
ops internas no grupo G:

$$(G ; \ op, \ inv, \ id)$$

$op : G \times G \to G$

$id : G$

$inv : G \to G$



ops externas ("constructions")

$G \times H$, $\quad G \oplus H$, $\quad \mathrm{Aut}(G)$, $\quad \ldots$

$A \times B$, $\quad A \cup B$, $\quad \wp A$, $\quad \ldots$

Product (binário)

(n-ário?)

$n \geq 3$: fácil $n = 0$? $= 1$?

formation

product construction

$$\frac{A \text{ type} \qquad B \text{ type}}{A \times B \text{ type}} \; (\times)\text{-F}$$

$(\times)$ é uma construção binária entre tipos

Introduction

pairing

$$\frac{a : A \qquad b : B}{\langle a, b \rangle_{A\times B} : A \times B} \; (\times)\text{-I}$$

Construtor de habitantes de $A \times B$

Como seria uma construção nulária? Assim:

$$\frac{}{\mathbb{N} \,.\, \text{Type}} \; (\mathbb{N})\text{-F}$$

poderia ter escolhido algo menos cultural: pair a b

pair(a, b)

e botar $\langle x, y \rangle \overset{\text{sug}}{\equiv}$ pair x y

Elimination

projections

$$\frac{w : A \times B}{w.l : A}\ (\times)\text{-}E_{1/L} \qquad \frac{w : A \times B}{w.r : B}\ (\times)\text{-}E_{2/R}$$

notações alternativas: $w.1$, $w.2$ — $w.l$, $w.r$ — $\mathrm{outl}\ w$, $\mathrm{outr}\ w$ — $\pi_1\ w$, $\pi_2\ w$

Será que $A \times B = A$? $A \times B = \mathbb{N}$?

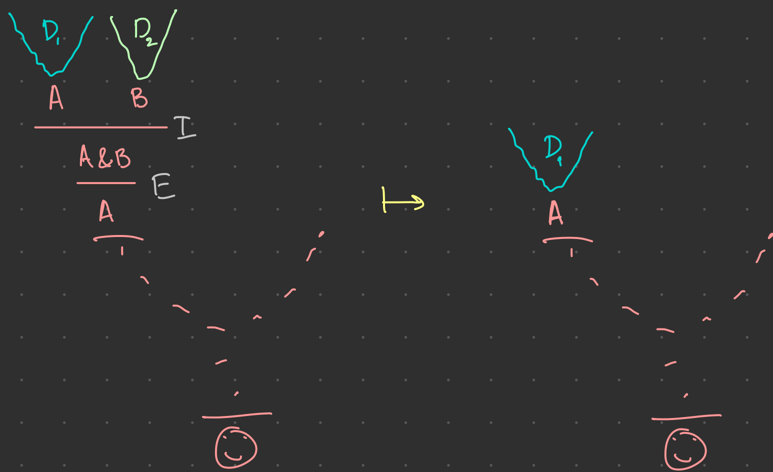
Equations

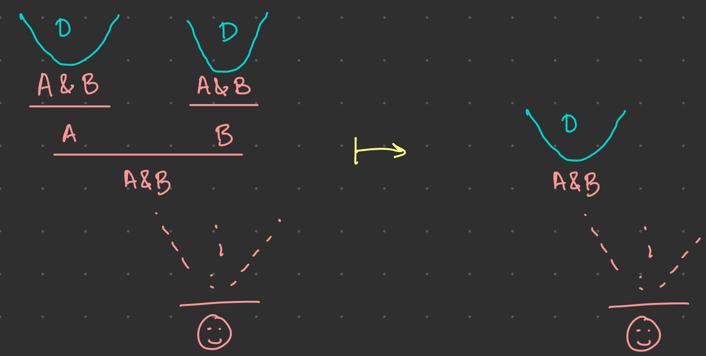
$$\langle a, b \rangle . l = a$$
$$\langle a, b \rangle . r = b \qquad (\beta)$$

$$(w.l, w.r) = w \qquad (\eta)$$

$$\frac{w : A \times B \qquad B\ \text{type} \qquad A\ \text{type}}{w.l : A}$$

Proofs compute!





Diagramas comutativos

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"', "(\cdot.l)"] \arrow[r, "\pi_2", "(\cdot.r)"'] & B \\
& C \arrow[ul, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[u, dashed, "\text{“}p\ !\text{”}"] &
\end{tikzcd}
$$

o diagrama comuta

$$\Updownarrow$$

$$\pi_1 \circ p = f$$
$$\pi_2 \circ p = g$$

$$\langle \pi_1 \circ p, \pi_2 \circ p \rangle$$
$$\|$$
$$p$$

$$
\begin{tikzcd}
A \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & B \arrow[r, "f"] & C
\end{tikzcd}
$$

$$f \circ i = f \circ j$$

(mas não necessariamente $i = j$)

Igualdade

lec05
2025-03-31

~~Definição.~~ Teorema?

Sejam $w, w' : A \times B$.

$$w =_{A\times B} w' \quad \cancel{\overset{\text{def}}{\iff}} \overset{?}{\iff} \quad w \cdot l =_A w' \cdot l \ \& \ w \cdot r =_B w' \cdot r$$

Demons. $(\Rightarrow)$.

$$\dfrac{\dfrac{}{w \cdot l = w \cdot l}\,\text{refl} \qquad w = w'}{w \cdot l = w' \cdot l}\,\text{subs}$$

similar

$$w \cdot r = w' \cdot r$$

$(\Leftarrow)$. Calc:

$$\begin{aligned} w &= \langle w \cdot l, w \cdot r \rangle && [(\eta)] \\ &= \langle w' \cdot l, w' \cdot r \rangle && [\text{hip} ; \text{hip}] \\ &= w' && [(\eta)] \end{aligned}$$

Categorias

Dados de uma categoria $\mathbb{C}$

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, bend right, "j"'] & C \arrow[l, bend right, "i"']
\end{tikzcd}
$$

$\mathbb{C}_0$ ou $Obj(\mathbb{C})$: $A, B, C, \ldots$ objetos

$\mathbb{C}_1$ ou $Arr(\mathbb{C})$: $f, g, h, \ldots$ setas

$src(f) = A$

$tgt(f) = B$

$src, tgt : Arr(\mathbb{C}) \longrightarrow Obj(\mathbb{C})$

source target

Esrevemos como sinônimos:

$$
\left\{
\begin{array}{l}
A \xrightarrow{f} B \\
f : A \longrightarrow B \\
f \in \mathbb{C}(A, B) \\
src(f) = A \ \& \ tgt(f) = B
\end{array}
\right.
$$

coleção das setas de A para B

Operações

Para qualquer objeto A:

$$\mathrm{id}_A : \mathbb{C}(A, A)$$

Para quaisquer objetos A, B, C:

$$\mathrm{comp}_{A,B,C} : \mathbb{C}(B, C) \times \mathbb{C}(A, B) \to \mathbb{C}(A, C)$$

Sinônimos:

$$\mathrm{id}_A \qquad 1_A$$

$$\mathrm{comp}_{A,B,C}(f, g) \qquad f \circ g \qquad g \,;\, f$$

Leis de categoria

(Ass) $(f \circ g) \circ h = f \circ (g \circ h)$

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[dr, "g \circ h"'] \arrow[drr, bend left=60, "(f \circ g) \circ h"] \arrow[drr, bend right=60, "f \circ (g \circ h)"'] & B \arrow[d, "g"] \arrow[dr, "f \circ g"] & \\
& C \arrow[r, "f"] & D
\end{tikzcd}$$

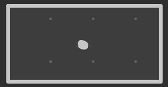
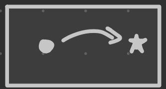
(Id) $id \circ f = f$

$f \circ id = f$

$$\begin{tikzcd}
A \arrow[r, "id"] \arrow[dr, bend right, "f"'] & A \arrow[d, "f"] \arrow[dr, bend left, "f"] & \\
& B \arrow[r, "id"'] & B
\end{tikzcd}$$

Exemplos

- Tipos, funções
- Conjuntos, funções
- Proposições, demonstrações
- Grupos, homomorfismos
- Posets, funções monótonas

- $\mathbb{0}$: 
- $\mathbb{1}$: 
- $\mathbb{2}$: 
- $\mathbb{3}$: ?

- Proposições, $(\vdash)$ $a \xrightarrow{f} b$ significa $a \vdash b$
- Inteiros, $(\mid)$ $a \xrightarrow{f} b$ significa $a \mid b$
- Inteiros, $(\leq)$ $a \xrightarrow{f} b$ significa $a \leq b$
- Conjuntos, $(\subseteq)$ $A \xrightarrow{f} B$ significa $A \subseteq B$

Universal properties : o produto

$$\forall \quad \begin{tikzcd} & & P \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & B \\ A & & & & \\ & & \forall C \arrow[uull, bend left, "\forall f"] \arrow[uurr, bend right, "\forall g"'] \arrow[uu, dashed, "\exists! p"] & & \end{tikzcd}$$

$? : C \longrightarrow A \times B$

$\langle f, g \rangle_C$

$\xleftarrow{\pi_1} P \xrightarrow{\pi_2}$ é um produto dos A, B
sse possui tal propriedade universal

Dualidade: coconceito

Coproduto

$$\begin{tikzcd}
A \arrow[r, "\iota_1"'] \arrow[dr, bend right, "\forall f"'] & S \arrow[d, dotted, "\exists! p"] & B \arrow[l, "\iota_2"] \arrow[dl, bend left, "\forall g"] \\
& \forall C &
\end{tikzcd}$$

$\xrightarrow{\iota_1} S \xleftarrow{\iota_2}$ é um coproduto dos A, B

sse possui tal propriedade universal

Coproduto: como se manifesta ...

... nos inteiros com (|):

(i) $a \mid s$ & $b \mid s$

(ii) $(\forall c)[\, a \mid c \ \&\ b \mid c \Rightarrow s \mid c \,]$

$(\forall c \text{ mult. com. dos } a, b)\ [\, s \mid c \,]$

é o mmc!

...e nas outras cats?

Terminal & inicial

T terminal sse $\forall A \quad \exists! \quad A \xrightarrow{!} T$

I coterminal (inicial) sse $\forall A \quad \exists! \quad I \xrightarrow{!} A$

Isomorfismos

$$A \cong B \overset{\text{def}}{\iff} (\exists f : A \to B)[\, f \text{ iso} \,]$$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f'"] \end{tikzcd}$$

$$f \text{ iso} \overset{\text{def}}{\iff} (\exists f' : A \leftarrow B)[\, f' \circ f = id_A \;\&\; f \circ f' = id_B \,]$$

"Unicidade" (a menos de isomorfismo)

$$(\forall u, v \text{ legais})[\, u \cong v \,]$$

Como entrar vs como sair

$h : C \longrightarrow A \times B$

$h\ c \overset{\text{def}}{\equiv} \langle \underbrace{\dots c \dots}_{f\,c} , \underbrace{\dots c \dots}_{g\,c} \rangle$

$A \times B \longrightarrow C$

?

(nada!)

$f : C \to A$

$f\ c \overset{\text{def}}{\equiv} \underbrace{\dots c \dots}_{f\,c}$

$g : C \to B$

$g\ c \overset{\text{def}}{\equiv} \underbrace{\dots c \dots}_{g\,c}$

Construções de Cats

lec06

2025-04-02

$$\mathbb{C} \longmapsto \mathbb{C}^{op}$$

Prosets & Posets vs Cats

(preorder) (partial order)

$$(P\,;\,\leq)$$

$$(\leq) : P \times P \to \mathrm{Prop}$$

$$\begin{tikzcd}
 & d & \\
b \arrow[ur] & & c \arrow[ul] \\
 & a \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & d & \\
b \arrow[ur, bend left, "f"] \arrow[ur, "g"] \arrow[ur, bend right, "h"'] & & c \arrow[ul, "r"'] \\
 & a \arrow[ul, bend left, "i"] \arrow[ul, "j"'] \arrow[ur, "u"'] &
\end{tikzcd}$$

partial order: Refl, Trans, Antis

preorder: Refl, Trans

Refl : $x \leq x$

Trans : $x \leq y \;\&\; y \leq z \implies x \leq z$

Antis : $x \leq y \;\&\; y \leq x \implies x = y$

Sum (coproduct)

formation

$$\frac{A \ \text{type} \qquad B \ \text{type}}{A + B \ \text{type}} \ (+)\text{-F}$$

soma

Introduction

L-injection

$$\frac{a : A}{l_{A+B} \cdot a : A + B} \ (+)\text{-}I_1$$

inl a

R-injection

$$\frac{b : B}{r_{A+B} \cdot b : A + B} \ (+)\text{-}I_2$$

inr b

Elimination

contexto

$$\frac{s : A + B \qquad \Gamma, a : A \vdash e_1 : C \qquad \Gamma, b : B \vdash e_2 : C}{\Gamma \vdash (\text{case } s \text{ of } \mid l \cdot a \rightsquigarrow e_1 \,;\, r \cdot b \rightsquigarrow e_2) : C}\ (+)\text{-E}$$

case

$\ldots a \ldots : C \qquad \ldots b \ldots : C$

Equations

a e_1 tendo substituido todas as ocorrências livres de a por x

$$(\text{case } l \cdot x \text{ of } \mid l \cdot a \rightsquigarrow e_1 \,;\, r \cdot b \rightsquigarrow e_2) = \boxed{e_1[a := x]}$$

$$(\text{case } r \cdot y \text{ of } \mid l \cdot a \rightsquigarrow e_1 \,;\, r \cdot b \rightsquigarrow e_2) = e_2[b := y] \quad (\beta)$$

$$(\text{case } s \text{ of } \mid l \cdot a \rightsquigarrow l \cdot a \,;\, r \cdot b \rightsquigarrow r \cdot b) = s \quad (\eta)$$

Currying

lec07

2025-04-02

$$((\alpha \times \beta) \to \gamma) \underset{\text{uncurry}_2}{\overset{\text{curry}_2}{\cong}} (\alpha \to (\beta \to \gamma))$$

?

$$\text{curry } f \overset{\text{def}}{\equiv} \lambda a : \alpha .\ \lambda b : \beta .\ \boxed{f \langle a, b \rangle}$$

$$\dfrac{f : (\alpha \times \beta) \to \gamma \qquad \dfrac{a : \alpha \qquad b : \beta}{\langle a, b \rangle : \alpha \times \beta}\,(\times)\text{-I}}{f \langle a, b \rangle : \gamma}\,(\to)\text{-E}$$

$b : \beta$

$a : \alpha$

$f : \alpha \times \beta \to \gamma$

ou:

$$\text{curry } f\ a\ b \overset{\text{def}}{\equiv} f \langle a, b \rangle$$

curry_n , $\mathrm{uncurry}_n$: « Similar. » ? « etc. » ?

$\mathrm{curry}_4\ f\ a\ b\ c\ d \overset{\mathrm{def}}{\equiv} f\ (a,b,c,d)$

$$\mathrm{curry}_3 : \boxed{A \times B \times C} \longrightarrow D$$

$$\mathrm{curry}_3 : (A \times B) \times C \longrightarrow D$$

$$\Big\downarrow \mathrm{curry}_2$$

$$(A \times B) \to (C \to D)$$

$$\Big\downarrow \mathrm{curry}_2$$

$$A \to (B \to (C \to D))$$

$$\mathrm{curry}_{n+1} \overset{\mathrm{def}}{\equiv} (\mathrm{curry}_2)^{\circ n}$$

$$\mathrm{curry}_3 : A \times (B \times C) \longrightarrow D$$

$$\Big\downarrow \mathrm{curry}_2$$

$$A \to ((B \times C) \to D)$$

$$\Big\downarrow ?$$

$$A \to (B \to (C \to D))$$

$$\mathrm{curry}_{n+1} \overset{\mathrm{def}}{\equiv} ?$$

Unit

$$\frac{\text{nada}}{\mathsf{Unit}\ \mathsf{type}}\ \text{Unit-F}$$

$\mathbb{1}$ $\mathbf{1}$ 1

$$\frac{}{\star : \mathsf{Unit}}\ \text{Unit-I}$$

☺ $()$ $\langle\rangle$

(nenhuma) Unit-E

?

β, η ?

Bool

$$\frac{}{\mathsf{Bool}\ \mathsf{type}}\ \text{Bool-F}$$

$\mathbb{B}$

$$\frac{}{\mathit{ff} : \mathsf{Bool}}\ \text{Bool-I}_F \qquad \frac{}{\mathit{tt} : \mathsf{Bool}}\ \text{Bool-I}_T$$

$$\frac{b : \mathsf{Bool} \qquad y : C \qquad n : C}{\mathsf{if}\ b\ \mathsf{then}\ y\ \mathsf{else}\ n : C}\ \text{Bool-E}$$

(β)

$$\mathsf{if}\ \mathit{tt}\ \mathsf{then}\ y\ \mathsf{else}\ n = y$$

$$\mathsf{if}\ \mathit{ff}\ \mathsf{then}\ y\ \mathsf{else}\ n = n$$

(η)

$$\mathsf{if}\ b\ \mathsf{then}\ \mathit{tt}\ \mathsf{else}\ \mathit{ff} = b$$

$$
\begin{tikzcd}
\mathrm{BinProd} \arrow[r, "\text{nularizar}"] \arrow[d, "\text{dual}"'] & \mathrm{Unit} \arrow[d, "\text{dual}"] \\
\mathrm{Bin\ Sum} \arrow[r, "\text{nularizar}"] & \boxed{\begin{matrix}\mathrm{Counit} \\ \shortparallel \\ \mathrm{Empty}\end{matrix}}
\end{tikzcd}
$$

Empty

$$\frac{}{\mathrm{Empty}\ \ \mathrm{type}}\ F$$

$\mathbb{0}$ $\mathbf{0}$ 0

Nenhuma-I

$$\frac{e : \mathrm{Empty}}{\mathrm{boom}_C\ e : C}\ E$$

abort_C

Quem é o void da C?

β, η ?

Funções de graça

Tendo... será que ganho ..

$$\begin{array}{c} C \\ \downarrow \\ C' \end{array} \qquad \begin{array}{c} \big(((A \times B) \to (C \to D)) \to (E + (D \to G))\big) \\ \downarrow ? \\ \big(((A \times B) \to (C' \to D)) \to (E + (D \to G))\big) \end{array}$$

$$\begin{array}{c} B \\ \downarrow \\ B' \end{array} \qquad \begin{array}{c} \big(((A \times B) \to (C \to D)) \to (E + (D \to G))\big) \\ \downarrow ? \\ \big(((A \times B') \to (C \to D)) \to (E + (D \to G))\big) \end{array}$$

$$\begin{array}{c} D \\ \downarrow \\ D' \end{array} \qquad \begin{array}{c} \big(((A \times B) \to (C \to D)) \to (E + (D \to G))\big) \\ \downarrow ? \\ \big(((A \times B) \to (C \to D')) \to (E + (D' \to G))\big) \end{array}$$

Quem é o void da C?

lec08

2025-04-09

```
int f(void)
    return 42;
```

```
void g(int x)
    return;
```

$$\frac{g : \mathrm{int} \to \mathrm{void} \qquad 42 : \mathrm{int}}{g(42) : \mathrm{void}}$$

$f(a_1, \dots, a_n)$

$f \,\text{␣}\, \langle a_1, \dots, a_n \rangle$

$f \,\text{␣}\, a_1 \,\text{␣}\, \dots \,\text{␣}\, a_n$

n-ária

n-ária descurrificada

n-ária currificada

$f()$ ← dá falsa impressão que f recebeu nada

$f \,\text{␣}\, ()$ ← deixa claro que f recebeu ()

o único habitante do $\mathbb{1}$

Function space

$$\frac{A\ \mathsf{type} \qquad B\ \mathsf{type}}{A \to B\ \mathsf{type}}\ (\to)\text{-F}$$

$$\frac{\Gamma, a : A \vdash b : B}{\Gamma \vdash \lambda a : A .\, b : A \to B}\ (\to)\text{-I}$$

$$\frac{f : A \to B \qquad a : A}{f \,\llcorner\!\lrcorner\, a : B}\ (\to)\text{-E}$$

β? η?

Exemplo:

$$\frac{w : \mathrm{Int},\ u : \mathrm{Int},\ a : \mathrm{Int} \vdash a + 2 \cdot u : \mathrm{Int}}{w : \mathrm{Int},\ u : \mathrm{Int} \vdash \lambda a : \alpha .\ a + 2 \cdot u : \mathrm{Int} \to \mathrm{Int}}$$

Regras de inferência

premissas (0 ou mais)

nome da regra (rótulo)

$$\frac{J_1 \quad \cdots \quad J_n}{J}\ \boxed{R}$$

juizos (judgements)

conclusão (exatamente 1)

Contexts

implícito : $a : A \vdash B$ type

$$a : A \ , \quad b : B \ , \quad c : C \ , \ \ldots$$

implícito: $\vdash A$ type

implícito: $a : A, \ b : B \vdash C$ type

$$a_1 : A_1 \ , \quad a_2 : A_2(a_1) \ , \quad a_3 : A_3(a_1, a_2) \ , \ \ldots$$

Escrevendo: $n : \mathbb{N} \ , \quad u : \mathbb{R}^n \ , \quad v : \mathbb{R}^n \ , \quad m : \mathbb{N} \ , \quad A : \mathbb{Z}^{n \times m}$

Implícitos: $\vdash \mathbb{N}$ type

$n : \mathbb{N} \vdash \mathbb{R}^n$ type

$n : \mathbb{N}, \ u : \mathbb{R}^n \vdash \mathbb{R}^n$ type

$n : \mathbb{N}, \ u : \mathbb{R}^n, \ v : \mathbb{R}^n \vdash \mathbb{N}$ type

$n : \mathbb{N}, \ u : \mathbb{R}^n, \ v : \mathbb{R}^n, \ m : \mathbb{N} \vdash \mathbb{Z}^{n \times m}$ type

Dependent types / Families

"$A : I \to \text{Type}$"

indexing

$i : I \vdash B(i)\ \text{type}$

$i : I \vdash B(i)\ \text{type}$

B é uma familia de tipos
(indexada por $i : I$)

B é um tipo dependente
(do $i : I$)

Exemplo:

"$\mathbb{R}^{\square} : \mathbb{N} \to \text{Type}$"

$n : \mathbb{N} \vdash \mathbb{R}^n\ \text{type}$

Exemplo:

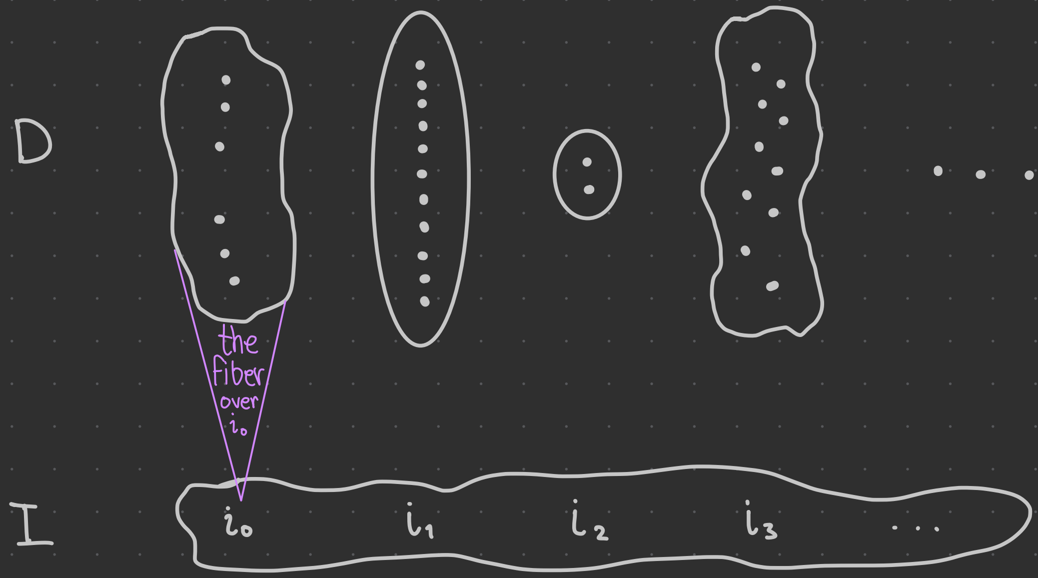
"$A : \mathbb{B} \to \text{Type}$"

$A(\text{ff}) \overset{\text{def}}{\equiv} \mathbb{N}$

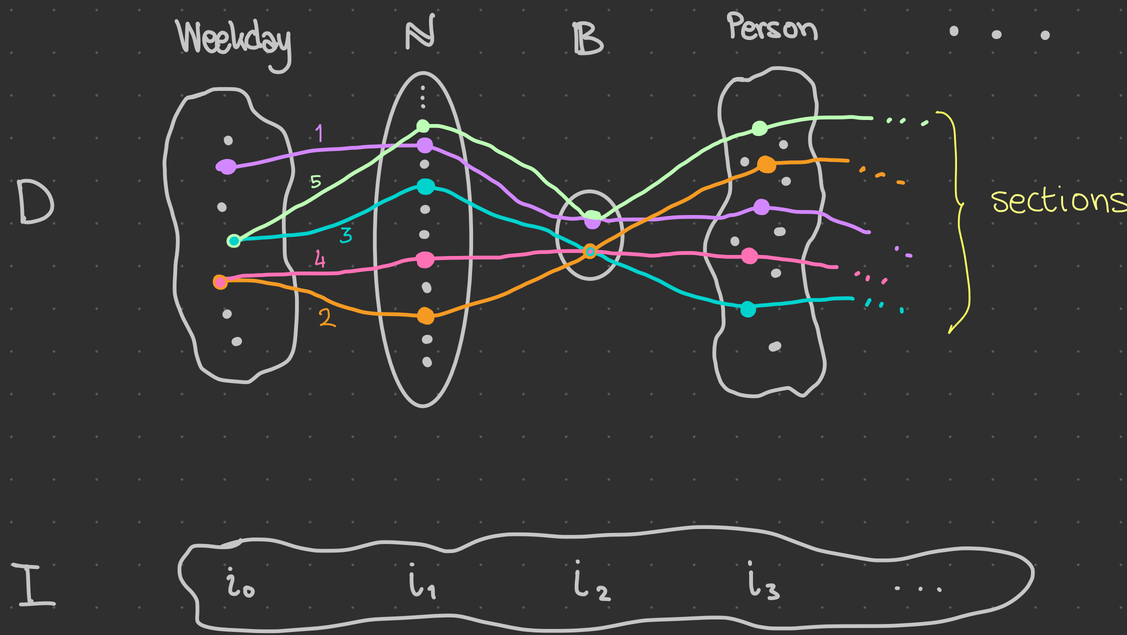
$A(\text{tt}) \overset{\text{def}}{\equiv} \mathbb{N} \times (\mathbb{N} \to \mathbb{N})$

$$\boxed{b : \mathbb{B} \vdash A(b)\ \text{type}}$$

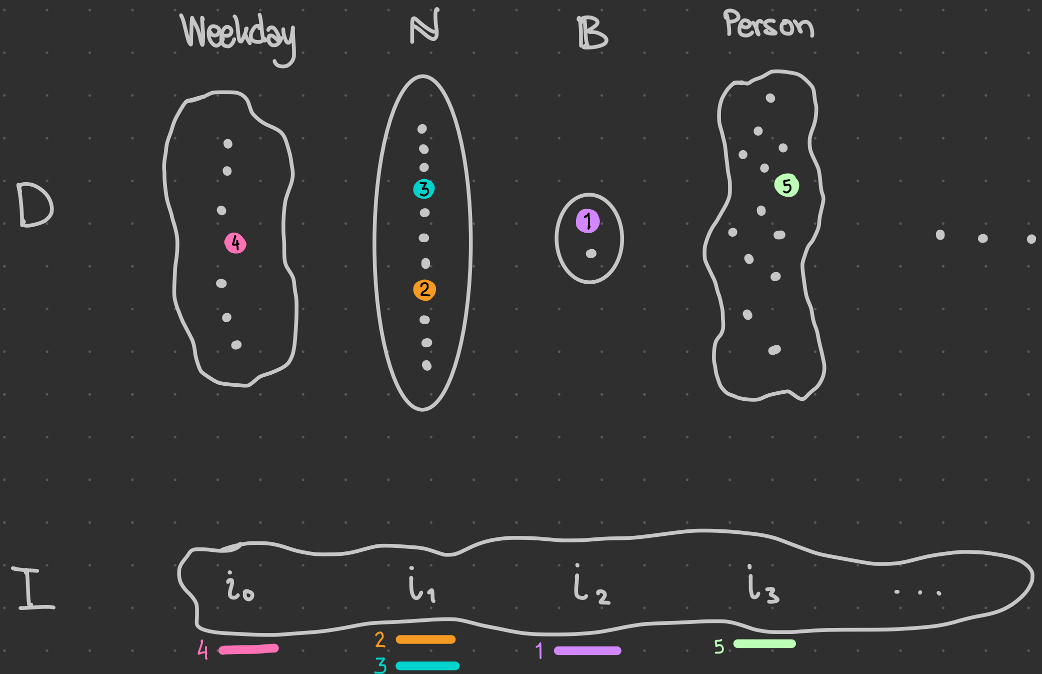
$$i : I \vdash D(i) \text{ type}$$



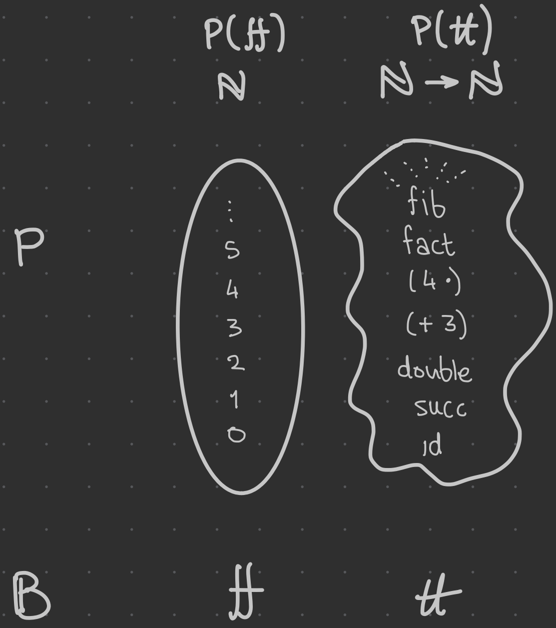
5 habitantes do ΠD



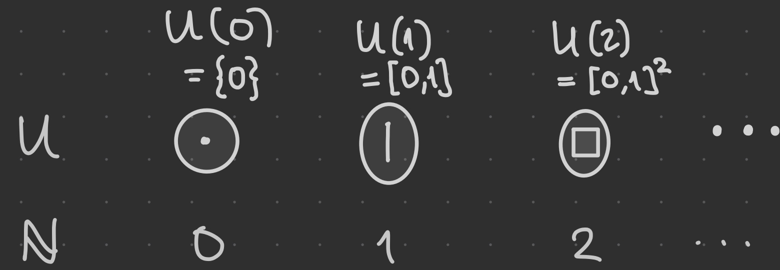
5 habitantes do ΣD



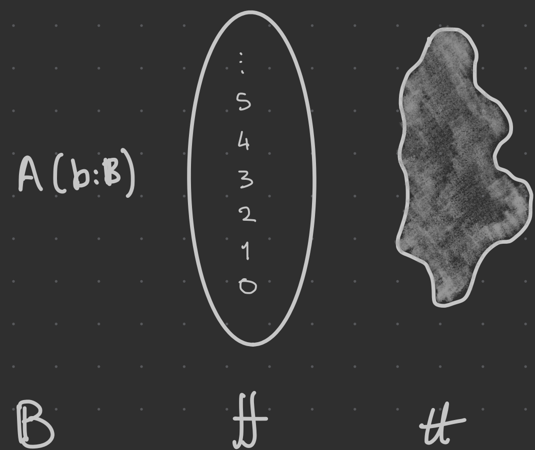
$$b : \mathbb{B} \vdash P(b) \ \text{type}$$



$$n : \mathbb{N} \vdash U(n) \ \text{type}$$



$$b : \mathbb{B} \vdash A(b) \ \text{type}$$



Π (Dependent product)

A é um dependent type

$$\dfrac{\boxed{i : I \vdash A(i) \ \text{type}}}{\vdash \prod_{i:I} A(i) \ \text{type}} \ (\Pi)\text{-F}$$

$$\Pi(i:I).A(i)$$

$$(i:I) \to A(i)$$

$$? : (b:\mathbb{B}) \to A(b)$$

I? E? β, η?

Σ (Dependent sum)

$$\frac{i : I \vdash A(i) \ \text{type}}{\vdash \sum_{i:I} A(i) \ \text{type}} \ (\Sigma)\text{-F}$$

$$\sum (i : I) . A(i)$$

$$(i : I) \times A(i)$$

$$? : (b : \mathbb{B}) \times A(b)$$

I? E? β, η?

Isomorfismos

lec09

2025-04-14

$$\alpha \cong \beta \overset{\text{def}}{\iff} (\exists f : \alpha \to \beta)[\ f \text{ iso}]$$

$$f : \alpha \to \beta \ \overset{\text{invertível}}{\text{iso}} \overset{\text{def}}{\iff} (\exists f' : \alpha \leftarrow \beta)\overbrace{[\ f' \circ f = \mathrm{id}_\alpha \ \& \ f \circ f' = \mathrm{id}_\beta\ ]}^{f' \text{ inversa de } f}$$

$$\alpha \xrightarrow{\cong} \beta$$

$$f : \alpha \cong \beta$$

$$f^{-1} \overset{\text{def}}{\equiv} \text{a inversa de } f$$

Θ. $(\cong)$ é uma relação de equivalência:

$(\cong)$ - refl

$(\cong)$ - trans

$(\cong)$ - sym

Aritmética de tipos

$$(\alpha + \beta) + \gamma \cong \alpha + (\beta + \gamma)$$

$$0 + \alpha \cong \alpha$$

$$\alpha + \beta \cong \beta + \alpha$$

$$(\alpha \times \beta) \times \gamma \cong \alpha \times (\beta \times \gamma)$$

$$1 \times \alpha \cong \alpha$$

$$\alpha \times \beta \cong \beta \times \alpha$$

Distributividades:

$$\delta \times (\alpha + \beta) \cong (\delta \times \alpha) + (\delta \times \beta)$$

$$\delta \times 0 \cong 0$$

$$\delta \times \sum_{i:I} \alpha_i \cong \sum_{i:I} (\delta \times \alpha_i)$$

Exemplo:

$$f : \delta \times (\alpha + \beta) \cong (\delta \times \alpha) + (\delta \times \beta)$$

$$f\, \langle d, l \cdot a \rangle \overset{\mathrm{def}}{\equiv} l \cdot \langle d, a \rangle$$

$$f\, \langle d, r \cdot b \rangle \overset{\mathrm{def}}{\equiv} r \cdot \langle d, b \rangle$$

Sem pattern-matching:

$$f\ w \overset{\mathrm{def}}{\equiv} \texttt{case}\ \ w \cdot r\ \ \texttt{of} \begin{cases} l\ a \rightsquigarrow l \cdot \langle w \cdot l, a \rangle \\ r \cdot b \rightsquigarrow r \cdot \langle w \cdot l, b \rangle \end{cases}$$

Conclusão: ❤ pattern-matching

Exponenciação

$$\alpha^{\beta+\gamma} \cong \alpha^{\beta} \times \alpha^{\gamma}$$

versão I-ária

$$\alpha^{\sum_{i:I} \beta_i} \cong \prod_{i:I} \alpha^{\beta_i}$$

versão nulária

$$\alpha^{\mathbb{0}} \cong \mathbb{1}$$

$$\alpha^{\beta\times\gamma} \underset{\text{currificação}}{\cong} (\alpha^{\beta})^{\gamma}$$

Conclusão: Aprendi currificação no ensino médio !?

$$\alpha \mathbin{?} (\beta + \gamma) \cong (\alpha \mathbin{?} \beta) \times (\alpha \mathbin{?} \gamma)$$

$$\alpha \mathbin{?} (\beta \times \gamma) \cong (\alpha \mathbin{?} \beta) \mathbin{?} \gamma$$

$$\boxed{(?) \equiv (\leftarrow)}$$

$$\alpha^{\beta} \overset{\text{def}}{\equiv} \beta \to \alpha \quad \textbf{ou} \quad \alpha \leftarrow \beta \quad \textbf{ou} \quad \alpha^{\swarrow \beta}$$

Cardinalidades

$$|\alpha \times \beta| = |\alpha| \cdot |\beta|$$

$$|\alpha + \beta| = |\alpha| + |\beta|$$

$$|\alpha \leftarrow \beta| = |\alpha|^{|\beta|}$$

$$\mathbb{2} \;\underset{\cong}{\overset{\overset{?}{\mathrm{def}}}{\equiv}}\; \mathbb{1} + \mathbb{1} \cong \mathbb{B}$$

$$\alpha^{\mathbb{0}} \cong \mathbb{1}$$

$$(\alpha + \beta)^{2} \cong ?$$

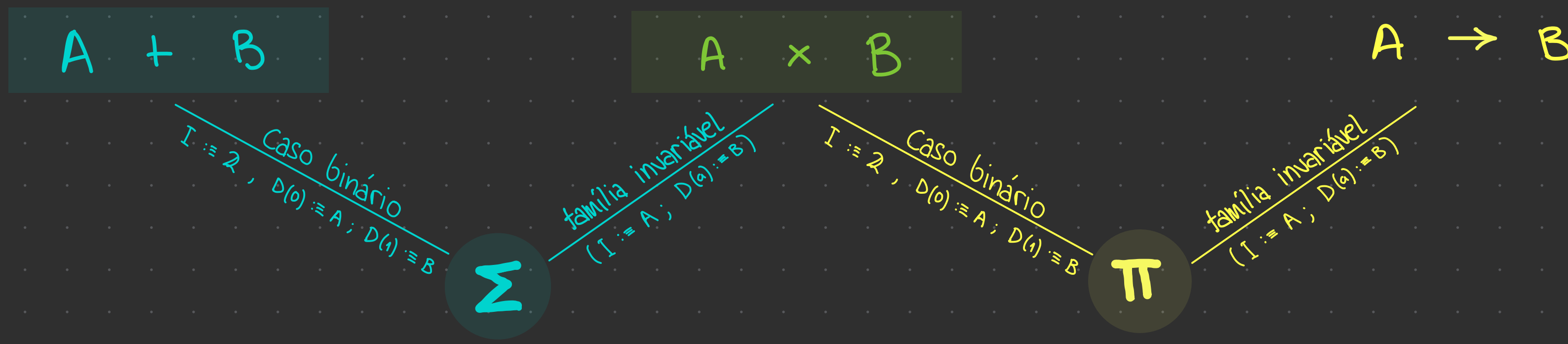
$$\alpha^{\mathbb{1}} \cong \alpha$$

$$3 \cong \mathbb{2} + \mathbb{1}$$

$$\alpha^{2} \cong \alpha \times \alpha$$

$$\vdots$$

$$n$$



família invariável: $I :\equiv A \;;\; D(a) :\equiv B$

$$\sum_{(a:A)} D(a) \equiv (a:A) \times B \equiv \boxed{A \times B}$$

$$\prod_{(a:A)} D(a) \equiv (a:A) \to B \equiv \boxed{A \to B}$$

caso binário: $I :\equiv \mathbf{2} \;;\; D(0) :\equiv A \;;\; D(1) :\equiv B$

$$\sum_{(i:\mathbf{2})} D(i) \cong D(0) + D(1) \equiv \boxed{A + B}$$

$$\prod_{(i:\mathbf{2})} D(i) \cong D(0) \times D(1) \equiv \boxed{A \times B}$$

Propositions as Types (Curry-Howard)

Certos tipos podem ser vistos como proposições e seus habitantes (as construções dos) como evidências - demonstrações.

Logic	Types
$\bot$	$\mathbb{0}$
$\top$	$\mathbb{1}$
$P \wedge Q$	$P \times Q$
$P \Rightarrow Q$	$P \to Q$
$P \vee Q$	$P + Q$
$\forall x \in A.\ P(x)$	$\prod(x:A).\ P(x)$
$\exists x \in A.\ P(x)$	$\sum(x:A).\ P(x)$
$u \overset{?}{=}_A v$	$Id_A(u,v)$

decisão

disjunção

construção

existencial

$$\neg P \equiv P \Rightarrow \bot$$

$$P \Leftrightarrow Q \equiv \ldots$$

truncamento

$$\| P \| \overset{def}{\equiv} \neg\neg P$$

as identificações dos $u, v : A$

Previously ... Funções de graça

lec10

2025-04-16

Tendo... será que ganho ..

$$C \to C'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B) \to (C' \to D)) \to (E + (D \to G)))
\end{tikzcd}$$

$$B \to B'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B') \to (C \to D)) \to (E + (D \to G)))
\end{tikzcd}$$

$$D \to D'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B) \to (C \to D')) \to (E + (D' \to G)))
\end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[d, "?"] \\
A' & A' \times B
\end{tikzcd}
\qquad (a, b) \mapsto (f\,a, b)
\qquad
\begin{tikzcd}
B \arrow[d, "g"] & A \times B \arrow[d, "?"] \\
B' & A \times B'
\end{tikzcd}
$$

Similar

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A + B \arrow[d, "?"] \\
A' & A' + B
\end{tikzcd}
\qquad
\begin{aligned}
(l \cdot a) &\mapsto l \cdot f\,a \\
(r \cdot b) &\mapsto r \cdot b
\end{aligned}
\qquad
\begin{tikzcd}
B \arrow[d, "g"] & A \times B \arrow[d, "?"] \\
B' & A \times B'
\end{tikzcd}
$$

Similar

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \rightarrow B \\
A' & A' \xrightarrow{u'} B \arrow[u, "?"] \arrow[u, "\cancel{?}"', leftarrow]
\end{tikzcd}
$$

pre comp.

$$(\circ f)$$

$$u' \mapsto \lambda a.\, u'(f\,a)$$

$$
\begin{tikzcd}
B \arrow[d, "g"'] & A \xrightarrow{u} B \arrow[d, "?"] \\
B' & A \rightarrow B'
\end{tikzcd}
$$

post-comp

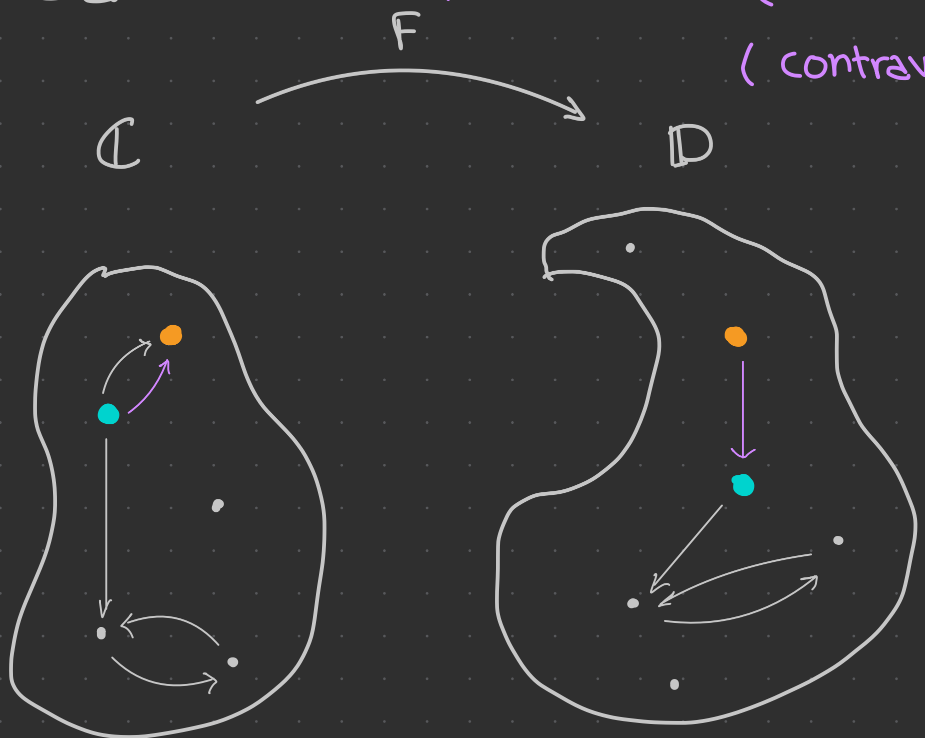
$$(g \circ)$$

$$u \mapsto \lambda a.\, \underbrace{g\,(u\,a)}_{(g \circ u)\,a}$$

Cats

functor (covariant)

(contravariant)



$$\mathrm{src} \circ F = F \circ \mathrm{src}$$

$$\mathrm{tgt} \circ F = F \circ \mathrm{tgt}$$

$$F(g \circ f) = Fg \circ Ff$$

$$F(id_A) = id_{FA}$$

$$F_0 : \mathbb{C}_0 \longrightarrow \mathbb{D}_0$$

$$F_1 : \mathbb{C}_1 \longrightarrow \mathbb{D}_1$$

comp. respeita functorialidade

hw

class Functor $F : Ty \to Ty$

$fmap : (A \xrightarrow{f} B) \xrightarrow{F} (F\,A \xrightarrow{F\,f} F\,B)$

(+ leis)

$$\mathrm{Int} \longmapsto \mathrm{List\ Int}$$

$$(\mathrm{Int} \to \mathrm{List\ Int}) \overset{\mathrm{List}}{\longmapsto} \mathrm{List}\ (\mathrm{Int} \to \mathrm{List\ Int})$$

(o Functor list sendo aplicado em objetos

instance Functor List $: Ty \to Ty$

$fmap : (A \to B) \to (\mathrm{List}\ A \to \mathrm{List}\ B)$

List

$\mathrm{List}_0 \quad : \quad \mathrm{Type} \to \mathrm{Type}$

$\mathrm{List}_1 \quad : \quad (A \to B) \to (L_0 A \to L_0 B)$

fmap

$$D \xmapsto{(A \times -)} A \times D$$

$$\begin{tikzcd}
D \arrow[d, "f"'] & & A \times D \arrow[d, "A \times f"] \\
D' \arrow[rr, phantom, "\xmapsto{(A \times -)}"] & & A \times D'
\end{tikzcd}$$

$(\times)$: covariante

$(+)$: covariante

$(\_ \to B)$: contravariante

$(A \to \_)$: covariante

Σ, Π

$$\frac{\Gamma \vdash I \ \text{type} \qquad \Gamma, i : I \vdash A_i \ \text{type}}{\Gamma \vdash (i : I) \times A_i \ \text{type}} \ \Sigma\text{-F}$$

ou $\sum_{i:I} A_i$ ou $\Sigma\, i : I .\, A_i$

$$\frac{\Gamma \vdash I \ \text{type} \qquad \Gamma, i : I \vdash A_i \ \text{type}}{\Gamma \vdash (i : I) \to A_i \ \text{type}} \ \Pi\text{-F}$$

ou $\prod_{i:I} A_i$ ou $\Pi\, i : I .\, A_i$

$$\frac{\Gamma \vdash i : I \qquad \Gamma \vdash a_i : A_i}{\Gamma \vdash \langle i, a_i \rangle : (i : I) \times A_i}\ \Sigma\text{-}I$$

$$\frac{\Gamma, i : I \vdash a_i : A_i}{\Gamma \vdash \lambda i : I.\ a_i : (i : I) \to A_i}\ \Pi\text{-}I$$

$$\frac{\Gamma \vdash w : (i : I) \times A_i}{\Gamma \vdash w.l : I \qquad \Gamma \vdash w.r : A_{w.l}}\ \Sigma\text{-}E$$

$$\frac{\Gamma \vdash f : (i : I) \to A_i \qquad \Gamma \vdash j : I}{\Gamma \vdash f \text{␣} j : A_j}\ \Pi\text{-}E$$

Universos

Universe : um tipo cujos elementos são tipos.

~~Type : Type~~ Girard

quão longa?

$\text{Type}_0 : \text{Type}_1 : \text{Type}_2 : \dots$

Galois connections

$$\begin{tikzcd} P \arrow[r, bend left, "F"] & Q \arrow[l, bend left, "G"] \end{tikzcd}$$

como comparar os p, q?

Quais são as setas?

Funções monótonas:

$$x \leq_P y \implies f\,x \leq_Q f\,y$$

Galois (connection)

$$F \dashv G \overset{\text{def}}{\iff} F\,p \leq_Q q \iff p \leq_P G\,q$$

lower ou left — adjoint — upper ou right

Θ, "dado um dos dois, o outro é determinado"

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "\text{ceil?}", "\text{floor?}"'] & \mathbb{Z} \arrow[l, bend left, "\text{inc}"] \end{tikzcd}$$

inc ☺ $\underline{?}$

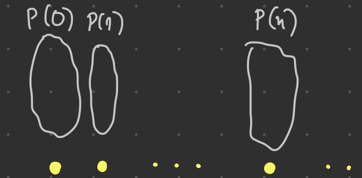
$\underline{?}$ ☺ inc

$\mathbb{N}$

$$\frac{}{\mathbb{N}\ \text{type}}\ \mathbb{N}\text{-F}$$

$$\frac{}{0 : \mathbb{N}}\ \mathbb{N}\text{-I}_z$$

$$\frac{n : \mathbb{N}}{\mathrm{succ}(n) : \mathbb{N}}\ \mathbb{N}\text{-I}_s$$



$$\frac{\begin{array}{l}\Gamma \vdash P : \mathbb{N} \to \mathrm{Type} \\ \Gamma \vdash b : P(0) \\ \Gamma,\ k : \mathbb{N},\ h : P(k) \vdash i(k,h) : P(\mathrm{succ}(k)) \\ \Gamma \vdash n : \mathbb{N}\end{array}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(P, b, i, n) : P(n)}\ \mathbb{N}\text{-E}$$

Equações:

$$\mathrm{ind}_{\mathbb{N}}(P, b, i, 0) = b \quad : P(0)$$

$$\mathrm{ind}_{\mathbb{N}}(P, b, i, \mathrm{succ}(k)) = i(k, \mathrm{ind}_{\mathbb{N}}(P, b, i, k)) : P(\mathrm{succ}(k))$$

MLTT

lec12

2025-04-28

Martin Löf Type Theory

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{\mathcal{C}}\ \text{label}$$

juízos

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash f : A \to B}{\Gamma \vdash f\ a : B}$$

(i) A é um tipo bem formada dentro do contexto Γ.

$$\Gamma \vdash A \text{ type}$$

judgementally equal

(ii) A, B são tipos juizamente iguais dentro do contexto Γ.

$$\Gamma \vdash A \doteq B \text{ type}$$

(iii) a é um elemento do tipo A no ctx Γ

$$\Gamma \vdash a : A$$

(iv) a, b são elementos jmente iguais do tipo A no Γ

$$\Gamma \vdash a \doteq b : A$$

$$\Gamma \vdash J$$

Contexto:

$$x_1 : A_1 ,\quad x_2 : A_2(x_1) ,\quad x_3 : A_3(x_1, x_2) ,\quad \ldots ,\quad x_n : A_n(x_1, \ldots, x_{n-1})$$

Para cada $1 \leq k \leq n$:

$$x_1 : A_1 , \ldots , x_{k-1} : A_{k-1}(x_1, \ldots, x_{k-2}) \vdash A_k(x_1, \ldots, x_{k-1}) \text{ type}$$

Nomes únicos!

Processo recursivo de verificação de ctx (de L para R).

Famílias de tipos

$$\Gamma, x : A \vdash B(x) \text{ type}$$

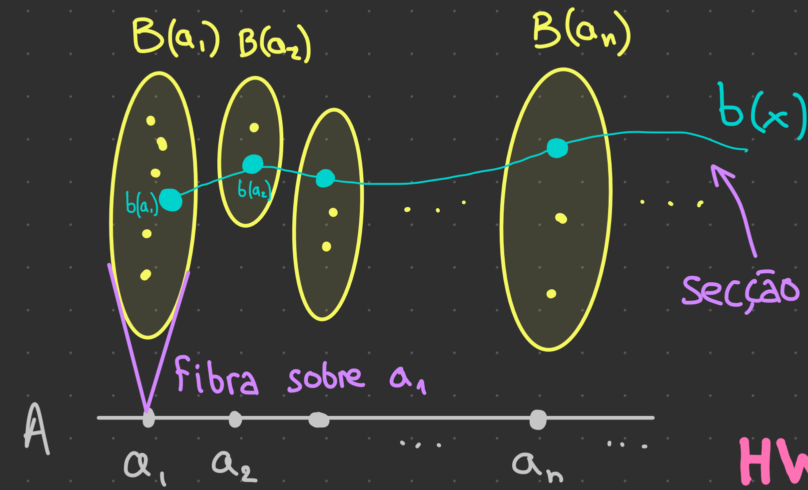
$A \to \text{Type}$

B é uma fam de tipos indexada por $x : A$

Exemplo:

$$n : \mathbb{N} \vdash \mathbb{R}^n \text{ type}$$

$$\Gamma, x : A \vdash b(x) : B(x)$$



HW

Regras

(i) Sobre info implícitas de formação de: ctx, tipos, elems:

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash A \text{ type}} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash A \text{ type}}$$

$$\frac{\Gamma \vdash A \doteq B \text{ type}}{\begin{array}{c}\Gamma \vdash A \text{ type} \\ \Gamma \vdash B \text{ type}\end{array}} \qquad \frac{\Gamma \vdash a \doteq b : A}{\begin{array}{c}\Gamma \vdash a : A \\ \Gamma \vdash b : A\end{array}}$$

? $\dfrac{\Gamma, x : A \vdash J}{\Gamma \vdash A \text{ type}}$ **HW**

inferível: $\dfrac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash A \text{ type}}$

$$\dfrac{\dfrac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash a : A}}{\Gamma \vdash A \text{ type}}$$

(ii) Que estabelecem que ($\doteq$) é uma rel de equivalência

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \doteq A \text{ type}}\ \text{refl-type} \qquad \textbf{refl} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash a \doteq a : A}\ \text{refl-elem}$$

$$\frac{\Gamma \vdash A \doteq B \text{ type} \quad \Gamma \vdash B \doteq C \text{ type}}{\Gamma \vdash A \doteq C \text{ type}} \qquad \textbf{trans} \qquad \frac{\Gamma \vdash a \doteq b : A \quad \Gamma \vdash b \doteq c : A}{\Gamma \vdash a \doteq c : A}$$

$$\frac{\Gamma \vdash A \doteq B \text{ type}}{\Gamma \vdash B \doteq A \text{ type}} \qquad \textbf{sym} \qquad \frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash b \doteq a : A}$$

(iii) conversão de variáveis

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, x : A', \Delta \vdash J}\ \mathrm{conv}(x : A') \quad (4\text{-in-}1)$$

(iv) Substituição $\quad e[a/x]$

$e[x := a] \overset{\text{def}}{\equiv}$ $\Big\{$ e substituindo todas as ocorrências de x por a tomando os devidos cuidados

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, \Delta[x := a] \vdash J[x := a]}\; S(x := a)$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, y_1 : B_1, \ldots, y_n : B_n \vdash C \text{ type}}{\Gamma, y_1 : B_1[x := a], \ldots, y_n : B_n[x := a] \vdash C[x := a] \text{ type}}$$

$$\frac{\Gamma \vdash a \doteq a' : A \qquad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[x := a] \vdash B[x := a] \doteq B[x := a'] \text{ type}}\; S(x := a, a')\text{-type}$$

$$\frac{\Gamma \vdash a \doteq a' : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[x := a] \vdash b[x := a] \doteq b[x := a'] : B[x := a]}\; S(x := a, a')\text{-elem}$$

(v) Weakening

lec13

2025-04-30

$$\frac{\Gamma \vdash A \ \text{type} \qquad \Gamma, \Delta \vdash J}{\Gamma, x : A, \Delta \vdash J} \ w(x:A) \ * \ x \text{ fresco}$$

CASO ESPECIAL:

$$\frac{\Gamma \vdash A \ \text{type} \qquad \Gamma \vdash B \ \text{type}}{\Gamma, x : A \vdash B \ \text{type}} \ w(x:A)$$

formação de família constante (trivialmente dependente)

(vi) ... finalmente:

$$\frac{\Gamma \vdash A \ \text{type}}{\Gamma, x : A \vdash x : A}\ \delta(x)$$

Derivações

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{C}$$

rename — HW

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, x' : A, \Delta[x := x'] \vdash J[x := x']} \ \mathsf{rename}(x, x') \ * \ x' \text{ fresco}$$

$$\dfrac{\dfrac{\Gamma \vdash A \ \mathsf{type}}{\Gamma, x' : A \vdash x' : A}\,\delta \qquad \dfrac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, x' : A, x : A, \Delta \vdash J}\,w\ (*)}{\Gamma, x' : A, \Delta[x := x'] \vdash J[x := x']}\,S$$

swap

$$\frac{\Gamma \vdash B \ \mathsf{type} \qquad \Gamma, x : A, y : B, \Delta \vdash J}{\Gamma, y : B, x : A, \Delta \vdash J}\ \mathrm{swap}(x,y)\ *$$

$$\dfrac{\dfrac{\Gamma \vdash B \ \mathsf{type}}{\Gamma, y : B \vdash y : B}\ \delta(y) \qquad \dfrac{\Gamma \vdash B \ \mathsf{type} \qquad \dfrac{\Gamma, x : A, y : B, \Delta \vdash J}{\Gamma, x : A, y' : B, \Delta[y := y'] \vdash J[y := y']}\ \mathrm{rename}(y, y')}{\Gamma, y : B, x : A, y' : B, \Delta[y := y'] \vdash J[y := y']}\ w(y)}{\Gamma, y : B, x : A, \underbrace{\Delta[y := y'][y' := y]}_{\Delta} \vdash \underbrace{J[y := y'][y' := y]}_{J}}\ S[y' := y]$$