Os racionais

lec01

2025-01-08

FTA : Int $\rightsquigarrow$ Rat

$$x : \text{Int} \xmapsto{\text{FTA}} p_1^{a_1} \cdots p_{n_x}^{a_{n_x}} : \text{List Nat}$$

$$r : \text{Rat} \xmapsto{\text{FTA}} p_1^{c_1} \cdots p_{n_r}^{c_{n_r}} : \text{List Int}$$

$$r = \frac{x}{y} = \frac{p_1^{a_1} \cdots p_{n_x}^{a_{n_x}}}{p_1^{b_1} \cdots p_{n_y}^{b_{n_y}}} = p_1^{a_1 - b_1} \cdots p_n^{a_n - b_n}$$

e.g. $\frac{24}{42} = \frac{2^3 \cdot 3^1 \cdot 5^0 \cdot 7^0}{2^1 \cdot 3^1 \cdot 5^0 \cdot 7^1} = 2^{3-1} \cdot 3^{1-1} \cdot 5^{0-0} \cdot 7^{0-1} = 2^2 \cdot 3^0 \cdot 5^0 \cdot 7^{-1}$

Potências

$$\frac{a : \mathrm{Int} \qquad n : \mathrm{Nat}}{a^n : \mathrm{Int}}$$

$$\frac{a : \mathrm{Rat} \qquad x : \mathrm{Int}}{a^x : \mathrm{Rat}}$$

e o 0^{-1} ?

Irracionais

Ache $x : ?$ t.q. $x^2 =_? 2 : ?$

Seja r : Rat t.q. $\boxed{r^2 =_{Rat} 2}$: Rat $\boxed{r^2 = 2} \rightsquigarrow \left(\frac{u}{v}\right)^2 = 2 \rightsquigarrow u^2 = 2v^2$

Sejam u, v : Int t.q. $u^2 =_{Int} 2v^2$: Int

Logo u^2 par.

Logo u par. [Lemma]

Logo seja k t.q $u = 2k$.

Logo $(2k)^2 = 2q^2$.

Logo $q^2 = 2k^2$.

Logo q^2 par.

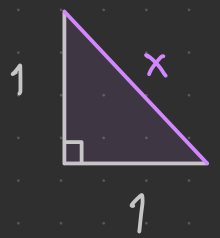
Logo q par. [mesmo Lemma]

Conclusão:

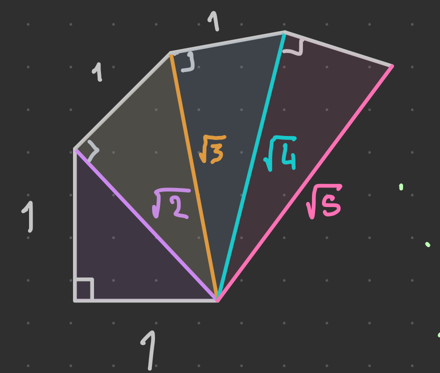
$u^2 = 2v^2 \implies u, v$ pares

...E agora? $\perp$?

A outra metade do argumento



$$x^2 = 1^2 + 1^2 = 2$$



5

$$\sqrt{-1} = \begin{cases} 3 \\ 2 \end{cases}$$

$$\sqrt{2}$$

6

Os reais (v1/?)

lec02
2025-01-10

$(\mathbb{R}\ ;\ 0, 1, +, -, \cdot\ )$

$0, 1 : \text{Real}$

$\_+\_, \_\cdot\_ : \text{Real} \times \text{Real} \to \text{Real}$

$\_^- : \text{Real} \to \text{Real}$

Axiomas:

$\mathbb{R}$A - ass	$\mathbb{R}$M - ass
- id	- id
- com	- com
- inv	- inv*

$(\forall a)[\, a \neq 0 \Rightarrow (\exists \ldots$

inv* : $(\forall a \neq 0)(\exists a')[\ a'$ é $(\cdot)$-inv de $a\ ]$

$\mathbb{R}$-dist : $\begin{cases} d(a+b) = da + db \\ (a+b)d = ad + bd \end{cases}$

Teoria dos reais (v1/?)

uni $\quad (+)\text{-id} \quad (+)\text{-inv}$

$\quad (\cdot)\text{-id} \quad (\cdot)\text{-inv}$

$(\forall \ell)(\forall \ell')[\ \ell, \ell' \text{ legais} \Rightarrow \ell = \ell'\ ]$

$(\forall \ell, \ell' \text{ legais})[\ \ell = \ell'\ ]$

res $\quad a + \square = b$

(*) $\quad a\ \square = b$

can $\quad a + c = b + c \Rightarrow a = b$

(*) $\quad a\ c = b \cdot c \Longrightarrow a = b$

ann $\quad a \cdot 0 = 0$

NZD $\quad ab = 0 \Rightarrow a = 0$ ou $b = 0$

Sugar

$$a - b \overset{\text{sug}}{\equiv} a + (-b)$$

$a^{-1} \overset{\text{sug}}{\equiv}$ o único $(\cdot)$-inv de a

$$a / b \overset{\text{sug}}{\equiv} a \cdot b^{-1}$$

$$a^{-n} \overset{\text{def}}{\equiv} \begin{cases} (a^n)^{-1} \\ \quad \parallel \ \Theta. \\ (a^{-1})^n \end{cases}$$

$$a^0 \equiv 1$$

$$a^{n+1} \equiv \left.\begin{cases} a^n \cdot a \\ a \cdot a^n \end{cases}\right\} \ \Theta \ \text{Tanto faz!}$$

Tipos dependentes (teaser)

$\_-\_ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

função parcial (pode não retornar nada)

$\_^{-1} : \mathbb{R} \rightharpoonup \mathbb{R}$

$\_^{-1} : ((a : \mathbb{R}) \times a \neq 0) \to \mathbb{R}$

$\_^{-1} : \mathbb{R} \times \underset{\substack{\text{Prop} \\ a \neq 0}}{?} \to \mathbb{R}$

quem é a?

$\_/\_ : (\mathbb{R} \times (b : \mathbb{R}) \times b \neq 0) \to \mathbb{R}$

"tuplas dependentes"

$$\left.\begin{array}{l} (3, (2, 0, 1)) \\ (2, (7, 7)) \\ (5, (3, 1, 1, 8, 3)) \end{array}\right\} : (n : \mathbb{N}) \times \mathbb{R}^n$$

"funções dependentes"

$f : (n : \mathbb{N}) \to \mathbb{R}^n$

$f\ 2 = (3, 1)$

$f\ 4 = (\sqrt{2}, 1, \pi, 3)$

Uma grande mentira

$\mathbb{N} \underset{\text{toInt}}{\subseteq} \mathbb{Z} \underset{\text{toRat}}{\subseteq} \mathbb{Q} \underset{\text{toReal}}{\subseteq} \mathbb{R} \underset{\text{toComplex}}{\subseteq} \mathbb{C}$

type errors !

$(\subseteq_\alpha) : \text{Set } \alpha \times \text{Set } \alpha \to \text{Prop}$

$3 : \text{Nat} \in \mathbb{Z}$

$(=_\alpha) : \alpha \times \alpha \to \text{Prop}$

type casting

$(\text{float})\ 3_{int} + 2.1_{float}$

$\downarrow$

3.0_{float}

type coertion

$3_{int} + 2.1_{float}$

$\downarrow$

$3.0_{float} + 2.1_{float}$

Conjuntos "sistemais" de reais

$$\mathbb{R}_\mathbb{N} \overset{\text{"def"}}{\equiv} \{0, 1, 2, \dots\} \quad : \text{Set Real}$$

$$\mathbb{R}_\mathbb{Z} \overset{\text{def}}{\equiv} \mathbb{R}_\mathbb{N} \cup -\mathbb{R}_\mathbb{N}$$

$$\mathbb{R}_\mathbb{Q} \overset{\text{def}}{\equiv} \{x/y \mid x \in \mathbb{R}_\mathbb{Z},\ y \in \mathbb{R}_\mathbb{Z} \setminus \{0\}\}$$

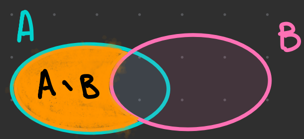
$$\mathbb{R}_\mathbb{A} \overset{\text{def}}{\equiv} \{x \mid x \text{ é raiz de algum polinômio no } \mathbb{Z}[x]\}$$

os reais algébricos

polinômios de uma variável com coeficientes inteiros

$$-A \overset{\text{def}}{\equiv} \{-a \mid a \in A\}$$

$$A \setminus B \overset{\text{def}}{\equiv} \{a \in A \mid a \notin B\}$$



Os reais (v2/?)

lec03
2025-01-15

$(\mathbb{R}\ ;\ +,\ 0,\ -,\ \cdot,\ 1,\ >)$

$\vdots$

$(>) : \mathrm{Real} \times \mathrm{Real} \to \mathrm{Prop}$

RO-Trans :	$a > b\ \&\ b > c \Rightarrow a > c$
-Tri	e.u.d.: $a > b\ ;\ a = b\ ;\ b > a$
-A	$a > b \Rightarrow a + c > b + c$
-M*	$a > b \Rightarrow c > 0 \Rightarrow ac > bc$

Sugar: $(\leq), (\geq), (<)$, Pos, Neg, ...

Teoremas!

Minima, maxima

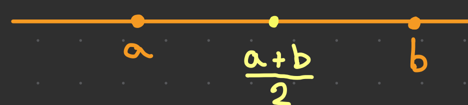
$\min, \max \;:\; \mathrm{Real} \times \mathrm{Real} \to \mathrm{Real}$

def

$$\min(a,b) = \frac{a+b-|b-a|}{2} = \frac{a+b}{2} - \frac{1}{2}d(a,b)$$

$$\max(a,b) = \frac{a+b+|b-a|}{2} = \frac{a+b}{2} + \frac{1}{2}d(a,b)$$

Teoremas!



Valor absoluto

Size

$$|\_| \; : \; \text{Real} \to \text{Real}$$

$$|x| \overset{\text{def}}{\equiv} \begin{cases} x, & \text{caso } x \geq 0; \\ -x, & \text{caso } x < 0. \end{cases}$$

Teoremas!

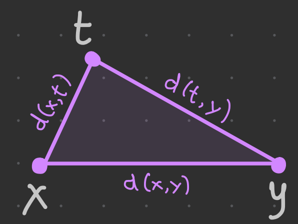
Distância

$$d : \text{Real} \times \text{Real} \to \text{Real}$$

melhor

$$\boxed{d : \alpha \times \alpha \to \text{Real}}$$

$$\text{pré-distância}\begin{cases} d(x,y) \geq 0 \\ d(x,y) \leq d(x,t) + d(t,y) & (\text{triangular}) \\ d(x,y) = d(y,x) \\ d(x,x) = 0 & \left(\Leftrightarrow \quad d(x,y) = 0 \Leftarrow x = y \right) \end{cases}$$



$$d(x,y) = 0 \Rightarrow x = y$$

Distâncias nos reais

Distância euclideana:

$$d(x, y) \overset{\text{def}}{\equiv} |x, y|$$

$\Theta.$ $d(x,y) \leq d(x,t) + d(t,y)$

$\vdots$

ALVO: $|x-y| \leq |x-t| + |t-y|$

$$\begin{aligned}
|x-y| &= |(x+0)-y| = |(x+(t+(-t))-y| \\
&= |x+t-t-y| \\
&= |(x-t)+(t-y)| \\
&\leq |x-t| + |t-y| \quad [\Theta.\!: |x+y| \leq |x|+|y|]
\end{aligned}$$

Verifique o resto!

Distância discreta:

$$d(x,y) \stackrel{\text{def}}{\equiv} \begin{cases} 0, & \text{caso } x = y; \\ 1, & \text{caso } x \neq y. \end{cases}$$

HW: ...é uma distância mesmo.

Conjuntos

lec04

2025-01-20

$\{0, 3.5\} : \text{Set Real}$

$\{\{0, \frac{1}{2}\}, \{42\}\} : \text{Set (Set Real)}$

$x \in A \overset{\text{def}}{\iff} ?$

$\{x \mid \underbrace{\ldots x \ldots}_{\varphi(x)}\} : \text{Set Real}$

$\varphi : \text{Real} \to \text{Prop}$

$\{w\, p \mid p \in P\} : \text{Set Real}$

$P : \text{Set Person}$

$w : \text{Person} \to \text{Real}$

Intervalos

$(a, b) \overset{\text{def}}{\equiv} \{ x \mid a < x < b \}$

$(a, b] \overset{\text{def}}{\equiv} \{ x \mid a < x \leq b \}$

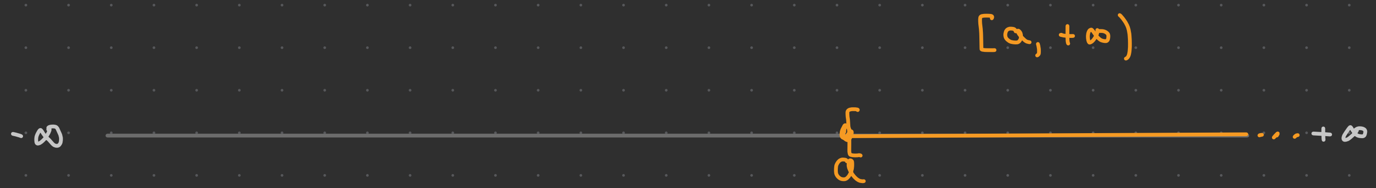
$[a, b) \overset{\text{def}}{\equiv} \{ x \mid a \leq x < b \}$

$[a, b] \overset{\text{def}}{\equiv} \{ x \mid a \leq x \leq b \}$

$(a, +\infty) \overset{\text{def}}{\equiv} \{ x \mid a < x \}$

$[a, +\infty), (-\infty, a), (-\infty, a]$

$(-\infty, +\infty) \overset{\text{def}}{\equiv} \mathbb{R}$



min, max

$$\mathrm{min},\ \mathrm{max} : \mathrm{Set\ Real} \overset{\rightharpoonup}{\to} \mathrm{Real}$$

$(500, +\infty)$ — Int: $\{501, 502, \ldots\}$; Real

$(1, 2)$ — $\emptyset$

$$\_ = \mathrm{min}\ \_ \quad : \quad \mathrm{Real} \times \mathrm{Set\ Real} \to \mathrm{Prop}$$
$$\_ = \mathrm{max}\ \_$$

$$x = \mathrm{min}\ A \overset{\mathrm{def}}{\iff} x \le A \ \&\ x \in A$$

$$x \le A \iff (\forall a \in A)[x \le a]$$

$$\mathrm{min}\ A < \mathrm{min}\ B \iff (\exists a)(\exists b)[\,a = \mathrm{min}\ A \ \&\ b = \mathrm{min}\ B \ \&\ a < b\,]$$

$$\mathrm{min}\ A + \mathrm{min}\ B < \mathrm{max}\ C \iff (\exists a, b, c)[\,a = \mathrm{min}\ A \ \& \ \ldots\,]$$

Floors & ceilings

$\lfloor \_ \rfloor_{Int} : Real \to Int$ floor (piso)

$\lfloor \_ \rfloor_{Real} : Real \to Real$ ceiling (teto)

Seqüências

$(0, 1, 2, \dots) : \text{Seq Real}$

$(0, 1, 0, 1, \dots) : \text{Seq Real}$

$(x_n)_n \equiv x_0, x_1, x_2, \dots$

$(x_n)_{n\in\mathbb{N}} \qquad (x_n)_{n=0}^{\infty}$

$(\underbrace{n^2 \mid}\, n \in \mathbb{N})$

Roubaram o Seq?

$\text{Seq } \alpha \overset{\text{impl}}{\equiv} \text{Nat} \to \alpha$

$\text{Seq} : \text{Type} \to \text{Type}$

$\text{Set} : \text{Type} \to \text{Type}$

Seq (Seq Real)

$\text{Set (Seq (Set Real))}$

$\vdots$

Quantas perguntas a um $A : \text{Set } \alpha$?

Tantas quantos os habs de α.

Quantas perguntas a uma $(a_n)_n : \text{Seq } \alpha$?

"Uma para cada dia."

↳ Tantas quantos os habs de Nat

$$(x_n)_n =_{\text{Seq }\alpha} (y_n)_n \overset{\text{def}}{\iff} (\forall n : \text{Nat})[\, x_n = y_n \,]$$

Exemplos

$$\left(1/2^n\right)_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \ldots$$

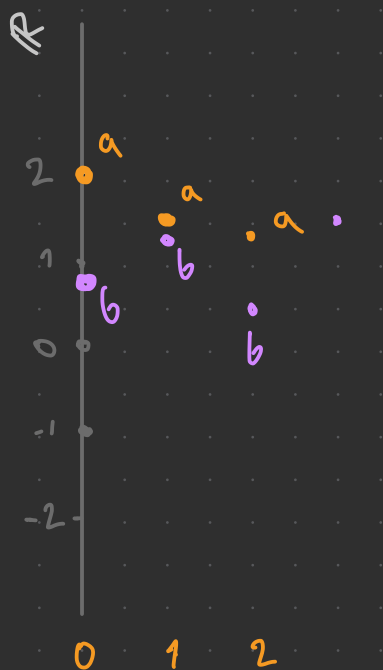
$$(1)_{n\in\mathbb{N}} = 1, 1, 1, \ldots$$

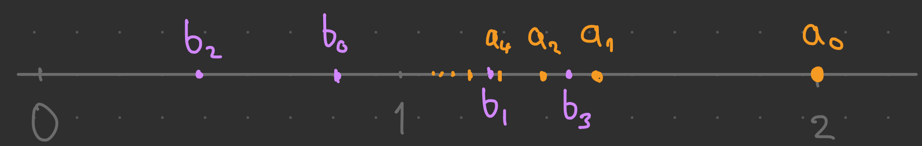
$$\left(\frac{n+1}{n}\right)_{n=2}^{\infty} = \frac{3}{2}, \frac{4}{3}, \ldots$$

$$b_n \overset{\text{def}}{\equiv} 1 + \frac{1}{n} \rightsquigarrow 2, 1+\frac{1}{2}, 1+\frac{1}{3}, \ldots$$

$$\begin{cases} a_0 \overset{\text{def}}{\equiv} 0 \\ a_{n+1} \overset{\text{def}}{\equiv} 1 - a_n \end{cases} \rightsquigarrow 0, 1, 0, 1, \ldots$$

Desenhando seqüências



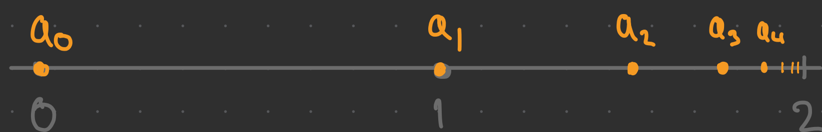


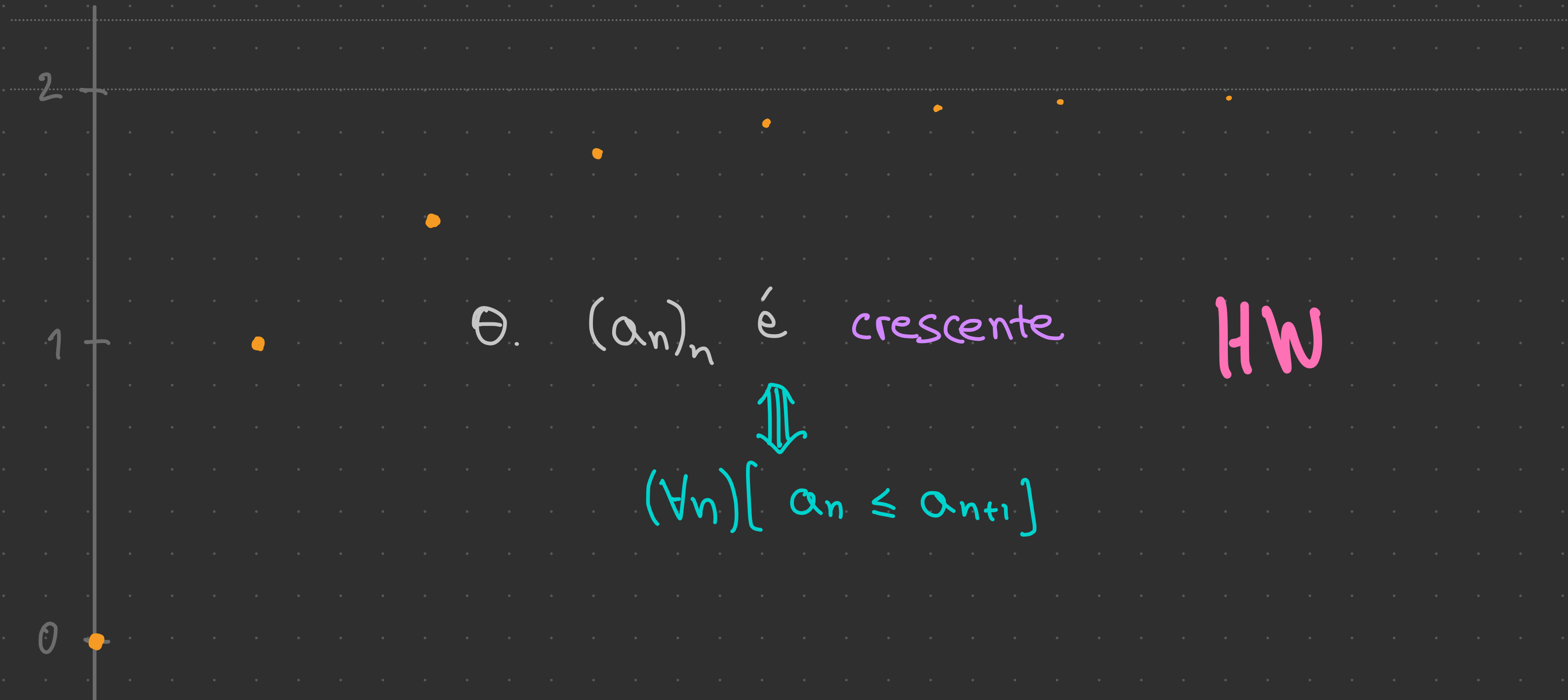
$$\begin{cases} a_0 \overset{\text{def}}{=} 0 \\ a_{n+1} \overset{\text{def}}{=} (1/2)\,a_n + 1 \end{cases}$$

$$0, \quad 1, \quad 1 + \frac{1}{2}, \quad 1 + \frac{3}{4}, \quad \dots$$

$$?, \quad 2 - \frac{1}{1}, \quad 2 - \frac{1}{2}, \quad 2 - \frac{1}{4}, \quad \dots$$

$$2 - \frac{1}{2^n}$$





Θ. $(a_n)_n$ é crescente HW

$$\Updownarrow$$

$$(\forall n)[\,a_n \le a_{n+1}\,]$$

Vocabulário

$(a_n)_n$ (estritamente) crescente / decrescente $\overset{\text{def}}{\iff}$ ...

$(a_n)_n$ superiormente / inferiormente cotada $\overset{\text{def}}{\iff}$ $(\exists c)[$ todos os a_n's $\leq$ / $\geq$ $c]$

$(\forall n)[\, a_n \leq c \,]$ / $\geq$

Investigar:

⋮

$(a_n)_n$ estritamente crescente $\overset{?}{\implies}$ $(a_n)_n$ não é sup. cot.

⋮

$\sup A \overset{\text{"def"}}{\equiv}$ a melhor cota superior de A

$A : \text{Set Real}$

$\sup (a_n)_n$ type error? Não pois type coerção:

$(a_n)_n \longmapsto \{a_n\}_n$

$\equiv$

$\{a_n \mid n \in \mathbb{N}\}$

$\inf A \overset{\text{"def"}}{\equiv}$ a melhor cota inferior de A

$\sup, \inf : \text{Set Real} \to \text{Real}$

$[0, 3)$

$\sup \{1, 2, 3\} = 3$

$\inf \{1, 2, 3\} = 1$

$\sup (0, \sqrt{2}) = \begin{cases} \sqrt{2} \\ \text{não tem} \end{cases}$

$\{ x : \text{Rat} \mid 0 < x^2 < 2 \}$

Supremum, infimum

lec05

2025-01-22

_ é um supremum de _ : Real → Set Real → Prop

$x = \sup A \overset{\text{def}}{\iff}$ x é uma cot. sup. de A & x é a melhor

$\Updownarrow$ $x \geq A$ $\Updownarrow$ $(\forall c \geq A)[x \leq c]$

$x = \inf A \overset{\text{def}}{\iff}$ ...

$\inf A < \sup B \iff$ ...

$$\boxed{A \text{ sup. cotado} \overset{?}{\Longrightarrow} A \text{ possui supremum}}$$

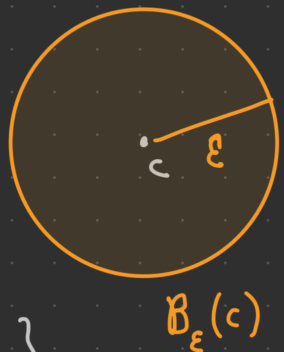
ε-perto, bolas

ε-vizinh

x ε-perto de y $\iff$ $d(x,y) \leq \varepsilon$

$\Updownarrow$

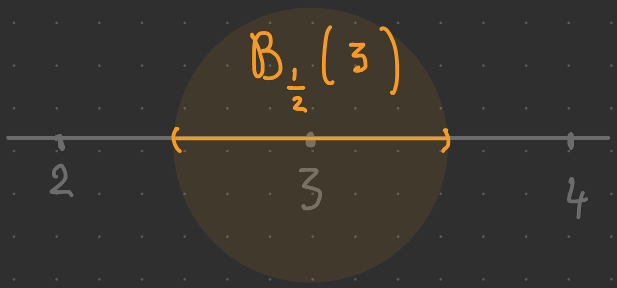
x,y estão ε-perto (entre si)
são



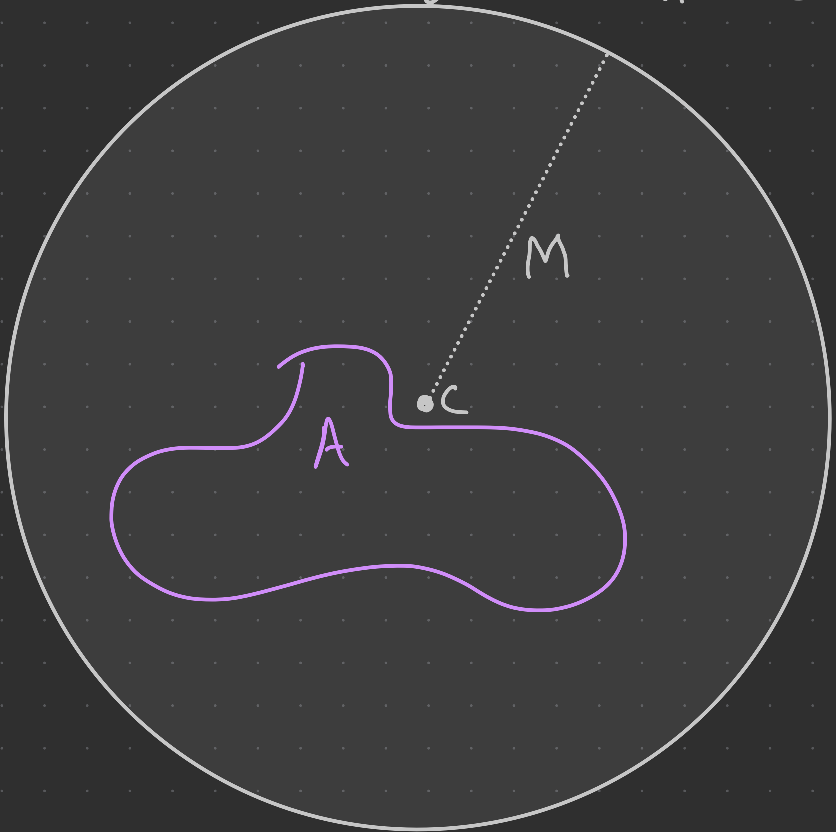
ε-bola de c $\overset{\text{def}}{\equiv}$ $\{\, x : \alpha \mid x$ ε-perto de $c\,\}$

$B_\varepsilon(c)$ $B_c(\varepsilon)$

$B(c;\varepsilon)$ $B(\varepsilon;c)$



A cercado $\overset{\text{def}}{\iff} (\exists c : \alpha)(\exists M)[\, A \subseteq B_M(c) \,]$



$(a_n)_n$ constante $\overset{\text{def}}{\iff}$ $(\exists k)(\forall n)[\, a_n = k \,]$

$(a_n)_n$ estável $\overset{\text{def}}{\iff}$ $(\forall n, m)[\, a_n = a_m \,]$

$(a_n)_n$ eventualmente linda

lindo : Real → Prop $\Updownarrow$ $\Updownarrow$ linda : Seq Real → Prop

$(\exists N)(\forall n \geq N)[\, a_n \text{ lindo} \,]$ $(\exists N)[\, (a_n)_{n \geq N} \text{ linda} \,]$

Limite de seqüência

$(a_n)_n \to \ell \overset{\text{def}}{\iff}$ qualquer bola no ℓ eventualmente contem a $(a_n)_n$

$\Updownarrow$

$\lim_n a_n = \ell$

$\Updownarrow$

$\lim (a_n)_n = \ell$

$\Updownarrow$

$(\forall \varepsilon > 0)[\, (a_n)_n$ eventualmente contida na $B_\varepsilon(\ell)\,]$

[.... fica ε-perto de ℓ]

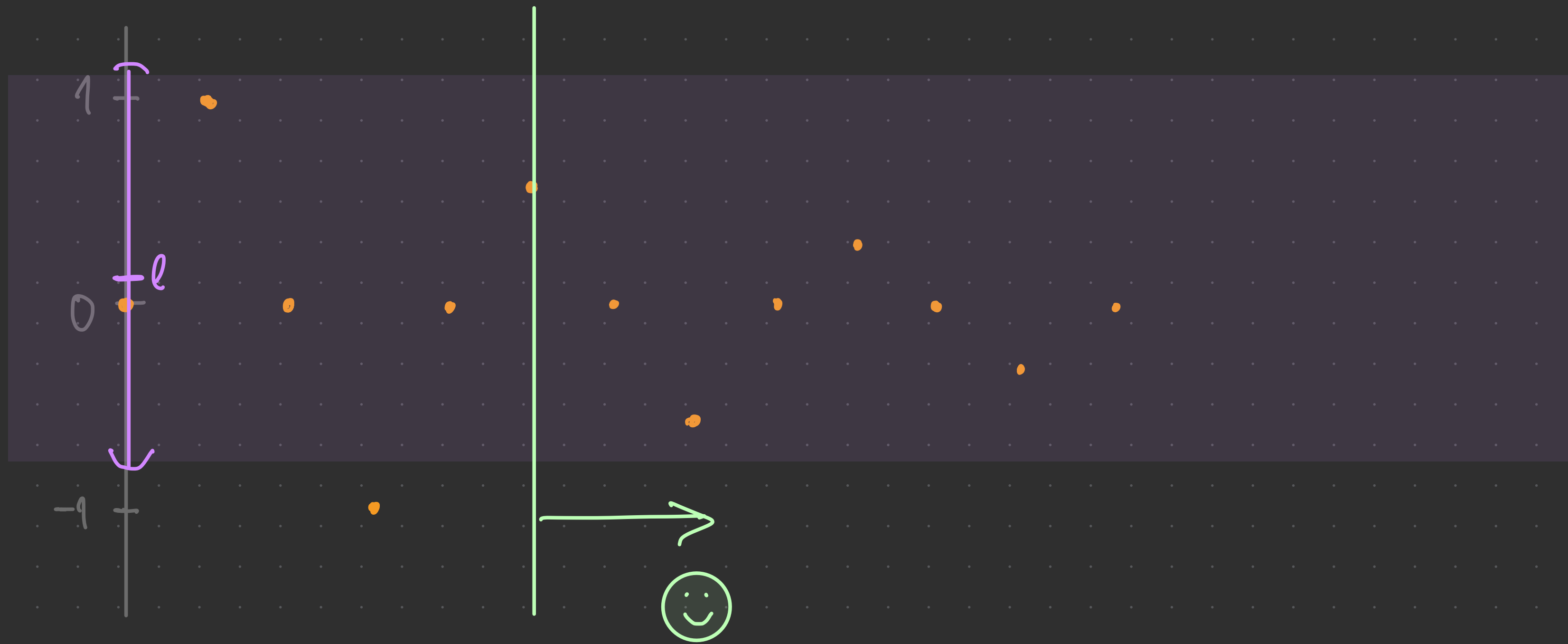
[para valores suficientemente grandes de n, a_n ε-perto de ℓ]

$(\exists N)[\, (a_n)_{n \geq N} \subseteq B_\varepsilon(\ell)\,]$

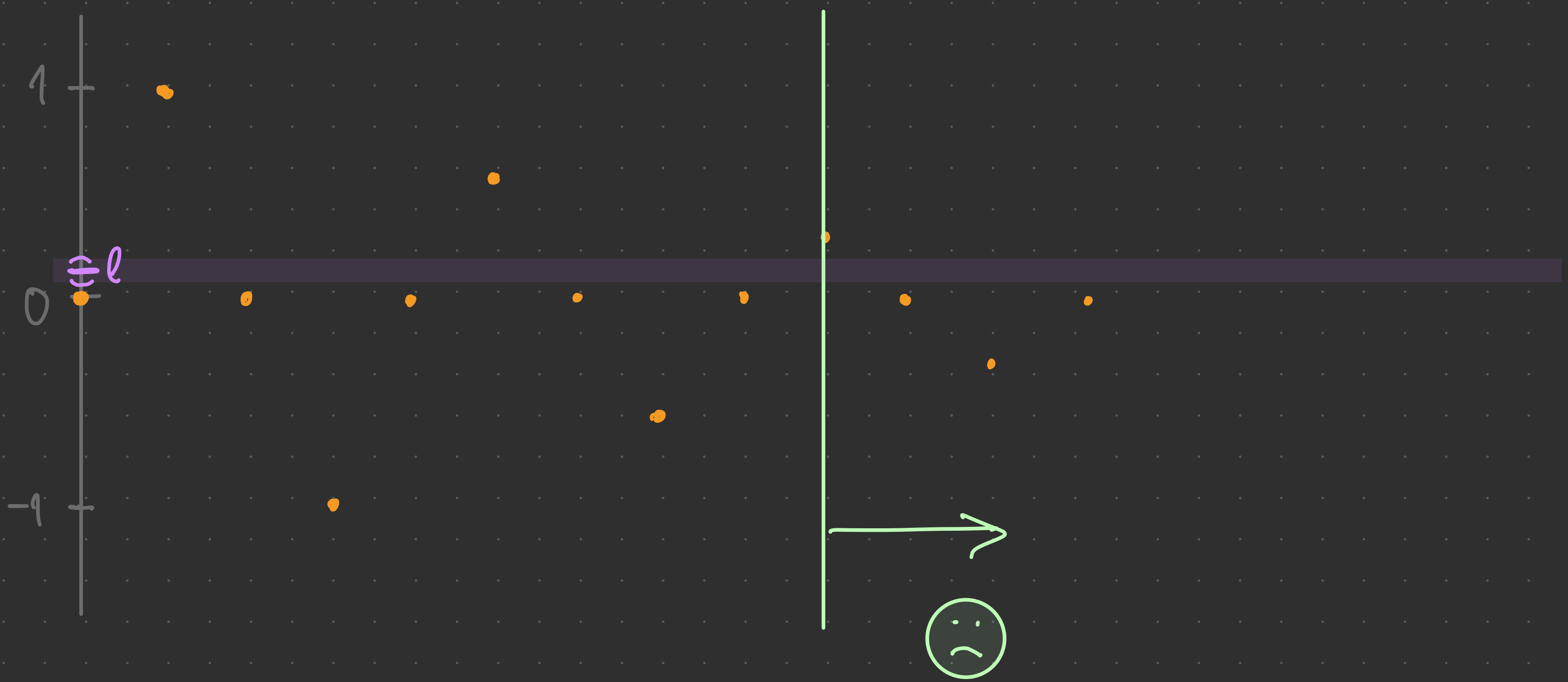
$(\forall n \geq N)[\, a_n \in B_\varepsilon(\ell)\,]$

$[\, d(a_n, \ell) < \varepsilon\,]$

$\_ \to \_ : \mathrm{Seq}\ \alpha \times \alpha \to \mathrm{Prop}$



1
0
-1
ℓ



1
0
-1
$\pm \ell$

$$\left.\begin{array}{c}\text{crescente} \\ \& \\ \text{sup. cot.}\end{array}\right\} \overset{?}{\Longrightarrow} \text{convergente}$$

$(a_n)_n$ autoconvergente $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall \varepsilon > 0)\,(\exists N)\,(\forall n, m > N)$

$[\, a_n,\ a_m \ \varepsilon\text{-perto} \,]$